Quasi-homomorphisms

This project is inspired by the following fact, which was discovered by Norbert A'Campo (*A natural construction for the real numbers*, arXiv: math/0301015). Denote by $\mathbf{Z}$ the additive group of integers. We call a map $f: \mathbf{Z} \to \mathbf{Z}$ an *almost homomorphism* if there exists an integer b such that for all $x, y \in \mathbf{Z}$ one has $|f(x+y) - f(x) - f(y)| \le b$, and we call two maps $g, h: \mathbf{Z} \to \mathbf{Z}$ *equivalent* if there exists an integer c such that for all $x \in \mathbf{Z}$ one has $|g(x) - h(x)| \le c$. One verifies without difficulty that the set of equivalence classes $[f]$ of almost homomorphisms $f: \mathbf{Z} \to \mathbf{Z}$ forms a *ring*, with addition induced by pointwise addition of functions, multiplication by composition of functions, and unit element $[\mathrm{id}_{\mathbf{Z}}]$. This ring is actually isomorphic to the field $\mathbf{R}$ of real numbers, which at first may sound surprising but is in fact not difficult to prove.

The proposed project is devoted to an algebraic generalization of the definitions just given.

Let A, B be abelian groups, which we shall write additively. By an *almost homomorphism* from A to B we mean a map $f: A \to B$ with the property that the subset $\{f(x+y) - f(x) - f(y) : x, y \in A\}$ of B is finite. The set $\mathrm{Ahom}(A, B)$ of almost homomorphisms $A \to B$ is an abelian group with pointwise addition: $(f+g)(x) = f(x) + g(x)$ for all $f, g \in \mathrm{Ahom}(A, B)$ and $x \in A$. It contains the set of "almost zero" maps $\mathrm{Az}(A, B) = \{h: A \to B : \#h(A) < \infty\}$ as a subgroup. By a *quasi-homomorphism* $A \to B$ we mean an element of the quotient group $\mathrm{Qhom}(A, B) = \mathrm{Ahom}(A, B)/\mathrm{Az}(A, B)$. For $f \in \mathrm{Ahom}(A, B)$, we shall denote the coset $f + \mathrm{Az}(A, B)$ by $[f]$; so $[f] \in \mathrm{Qhom}(A, B)$. Taking $A = B$ we obtain $\mathrm{Qend}(A) = \mathrm{Qhom}(A, A)$; it is not just an abelian group but even a *ring*, with multiplication $[f] \cdot [g] = [f \circ g]$ and unit element $[\mathrm{id}_A]$. The elements of $\mathrm{Qend}(A)$ are called *quasi-endomorphisms* of A. We can now formulate the result stated at the beginning by saying that, as a ring, $\mathrm{Qend}(\mathbf{Z})$ is isomorphic to the field $\mathbf{R}$ of real numbers.

Here is a list of questions that come to mind. Others may follow later!

Study the category **Qab** whose objects are all abelian groups, the set of morphisms from A to B being $\mathrm{Qhom}(A, B)$. Is it an additive category? What other good properties does it have?

The field $\mathbf{R}$ is, in number theory, viewed as just one of the "completions" of the field $\mathbf{Q}$ of rational numbers. Other completions are obtained by choosing a prime number p and viewing the field $\mathbf{Q}_p$ of *p-adic numbers*. Is $\mathbf{Q}_p$ also obtainable as $\mathrm{Qend}(A)$, for some abelian group A that depends on p? Which other fields can be obtained?

What does the ring $\mathrm{Qend}(\mathbf{Q})$ look like? Can one describe $\mathrm{Qhom}(\mathbf{Z}, A)$ for each abelian group A? If $\mathrm{Qhom}(A, B)$ is non-trivial, is it uncountable?

For many abelian groups A, the ring $\mathrm{Qend}(A)$ has an interesting topology. How generally can it be defined?

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